*April 1999*

# INTERMEDIATE MATHEMATICS LEAGUE OF EASTERN MASSACHUSETTS

# MEET #5

Category 1
Mystery
Meet #5 - April, 1999

*YOU MAY USE A*
*CALCULATOR*
*TODAY!*

1) **The houses on David's side of the street are numbered with consecutive even numbers. The first house on his side of the street is number 1456, and the last house is numbered 1672. What is the number of the 34th house on David's side of the street?**

2) **Find the difference between $\frac{3}{5}$ of 75% and $\frac{3}{4}$ of 60%.**

3) Definition: The digital root of a number is found by computing the sum of its digits, then computing the sum of the digits of that sum, and continuing to compute the sum of the digits until a one-digit number is obtained. For example, the digital sum of 2573 is found as follows: 2+5+7+3 = 17; 1+7 = 8. Therefore, the digital sum of 2573 is 8. Another example: The digital sum of $4^3$ (four cubed) is found like this: $4^3 = 64$; 6+4 = 10; 1+0 = 1. Therefore, the digital sum of $4^3$ is 1.
Find the digital sum of $2^{96}$. (two to the ninety-sixth power)

**ANSWERS**

1) _____

2) _____

3) _____

# SOLUTIONS - **Meet #5 - Category 1**

ANSWERS

**CATEGORY 1**

**MYSTERY**

**1) 1522**

**2) 0**

**3) 1**

1) This question is equivalent to "Find the 34th term of the sequence 1456, 1458, 1460, ..., 1672.

Analyzing the pattern should help:

| House | House number | Pattern |
|---|---|---|
| 1 | 1456 | 1456 + 2(0) |
| 2 | 1458 | 1456 + 2(1) |
| 3 | 1460 | 1456 + 2(2) |
| 4 | 1462 | 1456 + 2(3) |
| . | . | . |
| 34 | 1522 | 1456 + 2(33) |

Therefore, the 34th house on David's street is numbered **1522**.

2) The arithmetic for this question can be performed in a number of ways. The simplest may be to convert the percents into simple fractions:

$\frac{3}{4}$ of $\frac{3}{5}$ = $\frac{9}{20}$

$\frac{3}{5}$ of $\frac{3}{4}$ = $\frac{9}{20}$

The difference between these expressions is obviously **0**, which may be obvious without having done the multiplications!

# SOLUTIONS - **Meet #5 - Category 1**

3) Since $2^{96}$ is not really accessible via hand calculations, then looking for a pattern may be the only viable strategy:

| Power of 2 | Product | Digital Sum | Final |
|---|---|---|---|
| $2^0$ | 1 | 1 | 1 |
| $2^1$ | 2 | 2 | 2 |
| $2^2$ | 4 | 4 | 4 |
| $2^3$ | 8 | 8 | 8 |
| $2^4$ | 16 | 1+6=7 | 7 |
| $2^5$ | 32 | 3+2=5 | 5 |
| $2^6$ | 64 | 6+4=10; 1+0=0 | 1 |
| $2^7$ | 128 | 1+2+8=11; 1+1=2 | 2 |
| $2^8$ | 256 | 2+5+6=13; 1+3=4 | 4 |
| $2^9$ | 512 | 5+1+2=8 | 8 |
| $2^{10}$ | 1024 | 1+0+2+4=7 | 7 |
| $2^{11}$ | 2048 | 2+0+4+8=14; 1+4=5 | 5 |
| $2^{12}$ | 4096 | 4+0+9+6=19; 1+9=10; 1+0=1 | 1 |

The digital sums occur in a repeating pattern, so it is a matter of pairing up the powers of 2 with the digits of the digital sums:

| Exponent of power of 2: | **0** | 1 | 2 | 3 | 4 | 5 | **6** | 7 | 8 | 9 | 10 | 11 | **12** | 13 | 14 | 15 | 16 | 17 | **18** |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Digital Sum: | **1** | 2 | 4 | 8 | 7 | 5 | **1** | 2 | 4 | 8 | 7 | 5 | **1** | 2 | 4 | 8 | 7 | 5 | **1** |

Observation: Every sixth power of two has a digital sum of 1. Since $2^{96}$ is in that pattern of every sixth power (96 is a multiple of 6), its digital sum is **1**.

Category 2
Geometry
Meet #5 - April, 1999

*YOU MAY USE A*
*CALCULATOR*
*TODAY!*

1) **The sides of a cube are painted. The cube is then cut into 27 smaller cubes, all the same size. How many of the smaller cubes are painted on exactly two sides?**

2) **A basketball is packed tightly inside a closed cubical box so that the ball touches every surface of the box. If A:B represents the ratio of the volume of the basketball to the volume of the box, and A = π, then what is the value of B ?**

3) **It takes Fred twenty seconds to inflate a balloon in the shape of a sphere which is six inches in diameter. How many seconds would it take Fred to inflate a balloon which is eighteen inches in diameter if he inflates it at the same rate?**

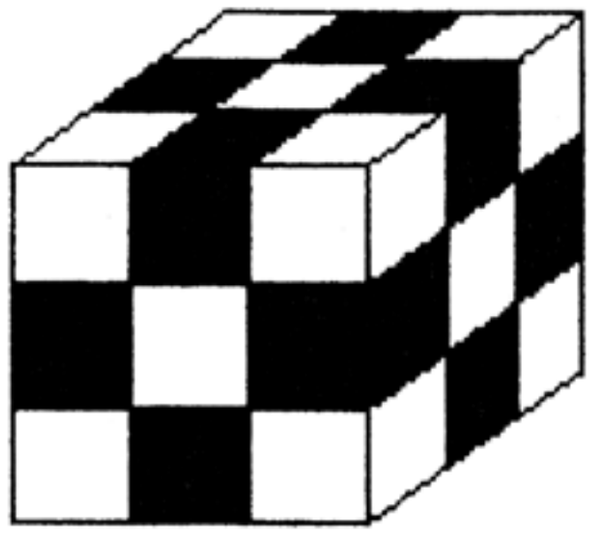
**ANSWERS**

1) _____

2) _____

3) _____

# SOLUTIONS - **Meet #5 - Category 2**



ANSWERS

**CATEGORY 2**
**GEOMETRY**

**1)** **12**

**2)** **6**

**3)** **540**

1) Of the 27 small cubes, **twelve** have only two sides which are painted. Some of them are highlighted in the diagram above. Imagine the rest.

2) $\frac{\text{volume of basketball}}{\text{volume of box}} = \frac{4/3\pi R^3}{LWH}$

$= \frac{4/3\pi R^3}{(2R)^3}$

$= \frac{4/3\pi R^3}{(2R)(2R)(2R)}$

$= \frac{4/3\pi R^3}{8R^3}$

$= \frac{4/3\pi}{8}$

$= \frac{4/3\pi}{8} \cdot \frac{3}{3} = \frac{4\pi}{24} = \frac{\pi}{6}$

Therefore, B = **6**.

3) If you multiply the diameter of a balloon by 3 (6·3 = 18), then you multiply its volume by $3^3$, or 27. Therefore, if it takes Fred 20 seconds to inflate the smaller balloon, then it would take him 27·20, or **540** seconds to inflate the larger balloon.

Category 3
Number Theory
Meet #5 - April, 1999

*YOU MAY USE A*
*CALCULATOR*
*TODAY!*

1) If $R = \{ 3, 6, 9, 12, 15, \ldots \}$
and $F = \{ 4, 8, 12, 16, \ldots \}$

then find the sum of the first four members of $R \cap F$.

2) If $A = \{ 1, 2, 3, 4, 5, 7, 11 \}$
and $A \cup B = \{ 1, 2, 3, 4, 5, 7, 9, 11, 13 \}$
and $A \cap B = \{ 1, 4, 5, 7 \}$

then find the sum of the members of set B.

3) The cast of The Wizard of Oz was polled, and it was discovered that 12 like pepperoni pizza, 10 like mushroom pizza, 4 like both, and 7 do not like either one. What percent of the cast like only mushroom pizza?

**ANSWERS**

1) _____

2) _____

3) _____

# SOLUTIONS - **Meet #5 - Category 3**

ANSWERS

**CATEGORY 3**
**NUMBER THEORY**

**1)** **120**

**2)** **39**

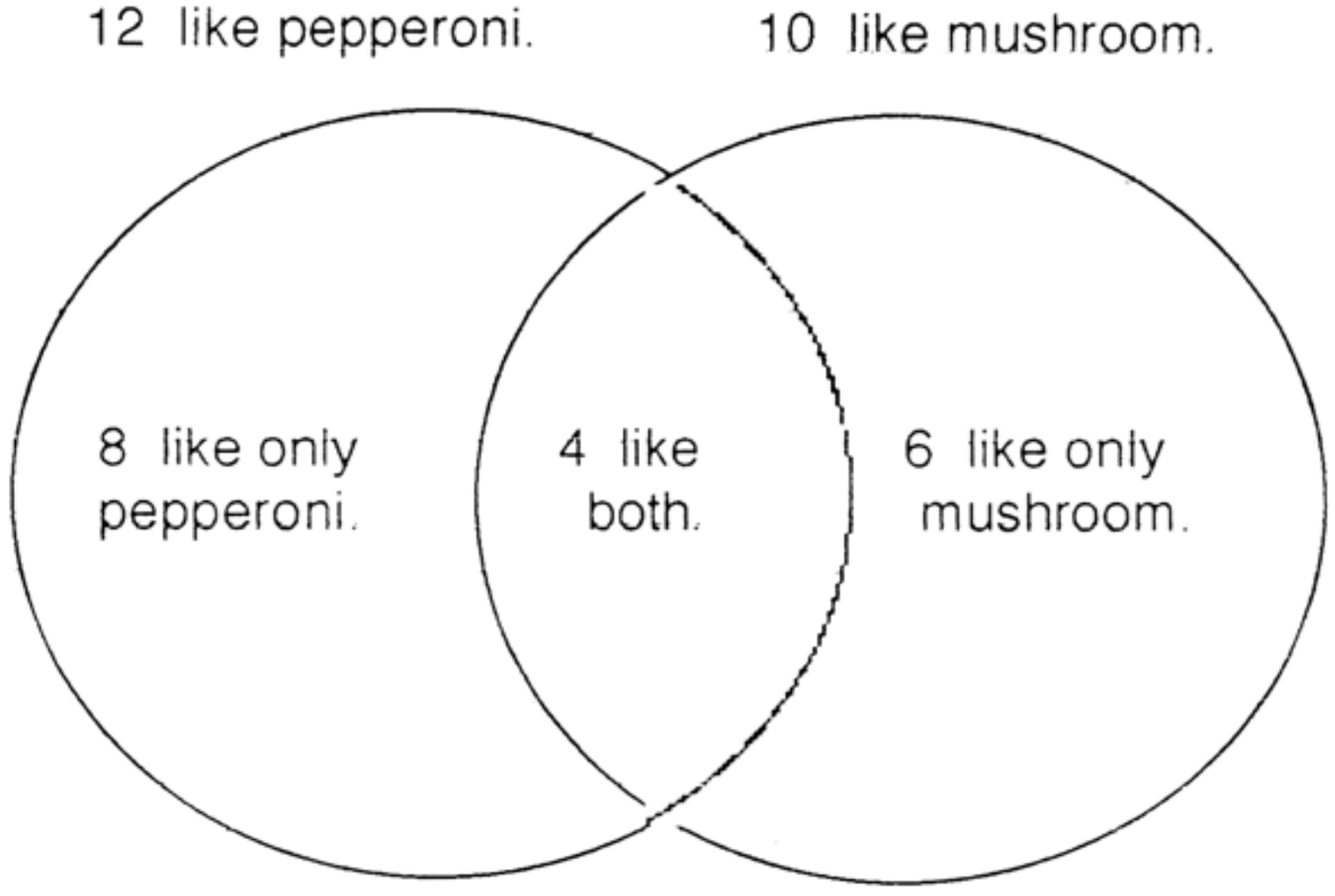
**3)** 24 (%)

1) This question is equivalent to "Find the sum of the first four common multiples of 3 and 4".

12 + 24 + 36 + 48 = **120**.

2) In order to satisfy the given conditions, set B must be { 1, 4, 5, 7, 9, 13 }, and the sum of its members is 1+4+5+7+9+13, or **39**.

3) The data in this problem can be represented in a Venn diagram as shown below:




The total number of cast members = 8+4+6+7, or 25. Therefore, the percent of the cast members who like only mushroom pizza is 6 ÷ 25 = 0.24, or **24%**.

Category 4
Arithmetic
Meet #5 - April, 1999

*YOU MAY USE A*
*CALCULATOR*
*TODAY!*

1) **Each surface of a cube is marked with either a å, ¥, or Ω. The cube is tossed and the top symbol is noted. In 450 tosses of the cube, the top surface was å 220 times, ¥ 83 times, and Ω 147 times. What is the most likely number of surfaces that contain the symbol å ?**

2) **What is the probability that a two-digit number, selected at random, is divisible by both 3 and 4 ? Express your answer as a simplified fraction.**

3) **Jim has a Fluffernutter for lunch 80% of the time. If he has a Fluffernutter for lunch, then there is a 90% probability that he will also have milk. If he does not have a Fluffernutter, then there is only a 35% probability that he will have milk. What is the probability that Jim will have milk with his lunch? Express your answer as a percent.**

**ANSWERS**

1) _____

2) _____

3) _____ %

# SOLUTIONS - **Meet #5 - Category 4**

ANSWERS

**CATEGORY 4**
**ARITHMETIC**

**1)** **3**

**2)** **$\frac{4}{45}$**

**3)** **79 (%)**

1) The symbol å appears 220 / 450 times, or about 48.9% of the time, which is practically 50%, or 1 / 2. Half of the six surfaces is **3**.

2) There are 90 two-digit numbers (all the whole numbers from 10 through 99, inclusive). The numbers which are divisible by both 3 and 4 are the multiples of 12, which are 12, 24, 36, 48, 60, 72, 84, and 96. There are eight of them. Therefore, theprobability of randomly selecting a two-digit number which is divisible by both 3 and 4 is 8 / 90, or **4 / 45**.

3) 90% (80%) + 35% (20%)
= 0.9 (0.8) + 0.35 (0.2)
= 0.72 + 0.07
= 0.79
= 79 / 100
= **79%**.

Category 5
Algebra
Meet #5 - April, 1999

*YOU MAY USE A*
*CALCULATOR*
*TODAY!*

1) **Find both values of N for which** $-12 + N^2 = -4N$.

2) **Two less than four times a number is seven less than its square. What is the sum of the two numbers which satisfy this condition?**

3) **The longer leg of a right triangle is eight inches less than the hypotenuse. The shorter leg is two inches more than 50% of the longer leg. What is the perimeter, in inches, of the triangle?**

**ANSWERS**

1) _____ , _____

2) _____

3) _____

# SOLUTIONS - **Meet #5 - Category 5**

ANSWERS

**CATEGORY 5**

**ALGEBRA**

**1)** **-6 , 2**
**(any order)**

**2)** **4**

**3)** **160**

1) $-12 + N^2 = -4N$

$N^2 + 4N - 12 = 0$

$(N+6)(N-2) = 0$

$N+6 = 0$ or $N-2 = 0$

$N = -6$ or $2$

2) $4N - 2 = N^2 - 7$

$N^2 - 4N - 5 = 0$

$(N-5)(N+1) = 0$

$N-5 = 0$ or $N+1 = 0$

$N = 5$ or $-1$

So, the sum of the solutions is $5 + (-1)$, or **4**.

3) Let $X$ = the length of the longer leg

$50\%X + 2$ = the length of the shorter leg

$X + 8$ = the length of the hypotenuse

Use the Pythagorean Theorem:

$(\text{leg})^2 + (\text{leg})^2 = (\text{hypotenuse})^2$

$(0.5X + 2)^2 + X^2 = (X + 8)^2$

$0.25X^2 + 2(2)(0.5)X + 4 + X^2 = X^2 + 16X + 64$

$0.25X^2 + 2X - 16X + 4 - 64 = 0$

$0.25X^2 - 14X - 60 = 0$

$X^2 - 56X - 240 = 0$

# SOLUTIONS - Meet #5 - Category 5

Using the quadratic formula may be easiest:

a = 1, b = -56, c = -240

$$X = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$X = \frac{-(-56) \pm \sqrt{(-56)^2 - 4(1)(-240)}}{2(1)}$$

$$X = \frac{56 \pm \sqrt{3136 + 960}}{2}$$

$$X = \frac{56 \pm \sqrt{4096}}{2}$$

$$X = \frac{56 \pm 64}{2}$$

$$X = \frac{56 + 64}{2} \quad \text{or} \quad X = \frac{56 - 64}{2}$$

$$X = \frac{120}{2} \quad \text{or} \quad X = \frac{-8}{2}$$

$$X = 60 \quad \text{or} \quad X = -4$$

Since the triangle can only have positive lengths, the longest leg (X) is 60, the shortest leg (50%X + 2) is 32, and the hypotenuse (X + 8) is 68.

Therefore, the perimeter of the triangle is 60 + 32 + 68, or **160** inches.

# Category 6
# Team Questions
# Meet #5 - April, 1999

1) **If three different odd positive integers are selected at random, and each is less than 50, then what is the probability that none of the three integers is a prime number?**

2) **The cube of a number is 3840 more than its square, and its square is 252 more than its square root. What is the value of that number?**

3) **One bag of stuffing will fill a "Tickle-Me-Elmo" doll which is 8 inches tall. How many bags of stuffing are required to fill a similar "Tickle-Me-Elmo" doll which is 32 inches tall?**

4) **Three sides of triangle ABC have integral (integer) lengths. Side AB is three feet longer than side BC. How many feet are in the least possible perimeter of triangle ABC?**

5) **How many degrees are in the acute angle made by the hands of a clock at 7:24 P.M.?**

6) **Let the answers to problems #1-5 be represented by the letters A, B, C, D, and E. Evaluate the following expression:**

**ANSWERS**

1) _____ = A

2) _____ = B

3) _____ = C

4) _____ = D

5) _____ = E

6) _____

$$\frac{C}{\left(E - \frac{1}{A} \cdot \frac{C+2}{B+\frac{C}{B}}\right)} - D$$

# SOLUTIONS - **Meet #5 - Category 6**

ANSWERS

**CATEGORY 6**
**TEAM QUESTIONS**

**1)** $\frac{33}{460}$

**2)** **16**

**3)** **64**

**4)** **9**

**5)** **78**

**6)** **- 7**

1) There are 25 odd positive integers which are less than 50. Of those, 14 are prime numbers. They are: 3,5,7,11,13,17,19,23,29,31,37,41,43,47. So, 11 of the 25 odd numbers are not prime. Therefore, the probability that none of the three integers is prime is

$$\frac{11}{25} X \frac{10}{24} X \frac{9}{23}$$

$$= \frac{990}{13{,}800}$$

$$= \frac{33}{460}$$

2) This question is far too complex to be solved with equations. "Guess and Check" is much quicker! Start at a reasonable point:

| $X$ | $\sqrt{X}$ | $X^2$ | $X^3$ | $X^3 - X^2$ | $X^2 - \sqrt{X}$ |
|---|---|---|---|---|---|
| 9 | 3 | 81 | 729 | 648 | 78 |
| **16** | 4 | 256 | 4096 | 3840 | 252 |

The number is **16**.

3) The linear ratio of the two dolls is 8:32, or 1:4. The volume ratio is the cube of the linear ratio, which is $(1:4)^3$, or 1:64. Therefore, **64** bags are required to fill the larger doll.

4) Two sides are already 1 and 4, which are the minimum lengths for sides AB and BC. To have a minimum perimeter, the third side could not be 1, because 1+1<4 (The sum of any two lengths must be longer than the third side.); could not be 2, because 1+2<4 (same reason); could not be 3, because 1+3=4 (same reason!).
Therefore, the minimum length for AC is 4.
The least perimeter is therefore 1+4+4, or **9**.

# SOLUTIONS - **Meet #5 - Category 6**

5) The minute marks on a clock are 6 degrees apart, and the hour marks are 30 degrees apart. At 7:24 P.M., the minute hand is pointing at the 24-minute mark, and the hour hand has rotated 24 / 60 of the way from the 7-hour mark toward the 8-hour mark.
24 / 60 of 30 degrees is (24 / 60)(30), or 12 degrees. So, going from the minute hand to the hour hand, in two stages:
from the minute hand to the 7-hour mark:
11 minutes, or 11(6) = 66 degrees.
from the 7-hour mark to the hour hand:
12 degrees.
total: 66 + 12 = **78** degrees.

6)
$$\frac{C}{\left(E - \frac{1}{A} \cdot \frac{C+2}{B + \frac{C}{B}}\right)} - D$$

$$= \frac{64}{\left(78 - \frac{1}{\frac{33}{460}} \cdot \frac{64+2}{16+\frac{64}{16}}\right)} - 9$$

$$= \frac{64}{\left(78 - \frac{460}{33} \cdot \frac{66}{16+4}\right)}$$

$$= \frac{64}{\left(78 - \frac{460}{33} \cdot \frac{66}{20}\right)} - 9$$

$$= \frac{64}{(78 - 46)} - 9$$

$$= \frac{64}{32} - 9 = 2 - 9 = -7$$