Sughrue

INTERMEDIATE MATHEMATICS LEAGUE OF EASTERN MASSACHUSETTS

MEET #4

Feb 1999

Category 1

Mystery

Meet #4 - February, 1999

YOU MAY USE A CALCULATOR TODAY!

1) **In the following addition problem, each letter represents a digit. What is the value of W - X + Y - Z ?**

$$\begin{array}{r} WWW \\ +\quad X \\ \hline YZZY \end{array}$$

2) **If N and P are positive integers, and $N^2 = P$, then P is said to be a perfect square. What percent of the natural numbers from 1 to 1000, inclusive, are perfect squares? Round your answer to the nearest whole percent.**

3) **What is the 127^{th} digit in the decimal representation of the fraction 4/7 ?**

ANSWERS

1) _____

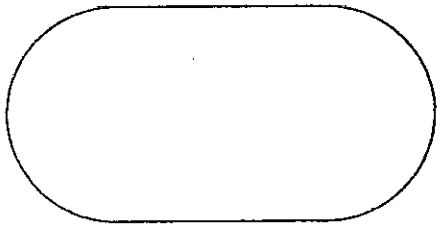
2) _____ %

3) _____

Category 2
Geometry
Meet #4 - February, 1999

1) If the circumference of a circle is 47.1 meters, then how many square meters are in its area? Use $\pi \approx 3.14$.

2) Jim runs four times around a track shaped like the one below. The track semicircles attached to either end of a rectangular field. The length of the rectangle is three times as long as its width. The radius of each semicircle is 15 meters. How many meters are in the total distance that Jim runs? Use $\pi \approx 3.1416$.



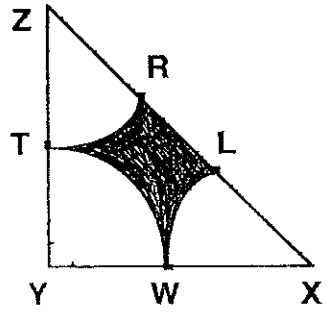
3) Refer to the figure below. XYZ is an isosceles right triangle, where angle XYZ is the right angle. Each leg is 10 centimeters long. W is the midpoint of XY, and T is the midpoint of YZ. Arcs WL, TW, and RT have centers at points X, Y, and Z, respectively. Find the number of square centimeters in the area of the shaded region. Use $\pi \approx 3.1$.

ANSWERS

1) _____ square meters

2) _____ meters

3) _____ square centimeters



Category 3
Number Theory
Meet #4 - February, 1999

1) **"Odie Day" was celebrated on the 312th day of Garfield's life. Garfield was born on a Tuesday. What day of the week was "Odie Day"?**

2) **Sam bought four Beanie Babies today. He will buy ten tomorrow, 16 the next day, 22 the following day, and so on. How many Beanie Babies will Sam buy on the 365^{th} day? (Note: Count today as the first day.)**

3) **What is the sum of the first 100 terms of the following sequence?**

4 11 18 25 32 39 . . .

ANSWERS

1) _____

2) _____

3) _____

Category 4
Arithmetic
Meet #4 - February, 1999

YOU MAY USE A
CALCULATOR
TODAY!

1) **Todd earns $2 per hour for each hour he works, plus 4% commission on the shoes he sells. How much did Todd earn on Saturday, if he sold 18 pairs of shoes at $45 per pair, and worked from 8:00 A.M. until 3:00 P.M.?**

2) **Harold and Maude are having lunch together. When the bill arrives, Harold wants to leave a tip which is 15% of the bill. Maude wants to leave a 20% tip, which is $0.90 more than what Harold wants to leave. How many dollars was the original bill?**

3) **Paul bought a new car, a Mazda Miata, for $16,990. He could not afford to pay cash, so he got a four-year loan from a local bank, which charged 9% interest per year, compounded once each year. What is the total amount that Paul paid for the car, including interest? Remember that rounding to the nearest hundredth of a dollar must occur after the calculation of interest each year. Your final answer must be rounded to the nearest hundredth of a dollar.**

ANSWERS

1) $ _____

2) $ _____

3) $ _____

Category 5

Algebra

Meet #4 - February, 1999

YOU MAY USE A CALCULATOR TODAY!

1) The sun was shining. The five-foot tall Jennifer measured her shadow on the ground, and it was 7.2 feet long. How many feet tall was a telephone pole whose shadow was 33.12 feet long?

2) Ken's 43-question history test was worth 232 points. The True-False questions were worth 6 points each, and the multiple-choice questions were worth 5 points each. How many multiple-choice questions were on the test?

3) If $\frac{N-2}{R-1} = \frac{3}{5}$ and $\frac{N-1}{R} = \frac{2}{3}$, then what is the value of the sum $(N + R)$?

ANSWERS

1) _____ feet

2) _____

3) _____

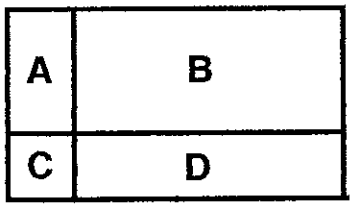
Category 6
Team Questions
Meet #4 - February, 1999

1) The degree measures of the angles of a triangle are three consecutive multiples of four. How many degrees are in the measure of the largest angle?

2) Ramanujan, the most famous Indian mathematician of the 20th century, discovered that the number 1729 is the smallest positive integer that can be expressed as the sum of two positive cubes in two different ways. If $M^3 + A^3 = 1729$, and $T^3 + H^3 = 1729$, then find the value of the sum $(M + A + T + H)$.

3) Every letter of our alphabet has been assigned a value, so A=1¢, B=2¢, ..., Z=26¢. How many letters are in the day of the week which is worth $1.00, when the values of its letters are added?

4) A rectangular floor of a house is separated into four smaller rectangular areas, as shown in the diagram below. The area of rectangle D is 48 square yards, the area of square C is 16 square yards, and the area of rectangle B is 60 square yards. What is number of yards in the perimeter of the original floor?



5) What is the sum of all positive integers, H, such that $10 \le H \le 100$, and H has exactly five positive integral factors?

6) Let the answers to problems #1-5 be represented by the letters A, B, C, D, and E. Evaluate the following expression:

ANSWERS

1) _____ = A

2) _____ = B

3) _____ = C

4) _____ = D

5) _____ = E

6) _____

$$\sqrt[4]{\frac{B}{2}}\sqrt{B^2-(E-A)^2-D+C+187}$$

SOLUTIONS - Meet #4 - Category 1

ANSWERS

CATEGORY 1
MYSTERY

1) 8

2) 3

3) 5

1) Because a single-digit number is being added, there are unique properties to this addition problem. The digit Y must be a 1, due to carrying, so W must be 9. X must then be 2 because, in the unit's column, W + X = the unit's digit of Y, or, 9 + X = 11. Z must be 0, again due to carrying. The calculation then looks like this:

$$\begin{array}{r} 999 \\ +\ \ 2 \\ \hline 1001 \end{array}$$

So, W - X + Y - Z = 9 - 2 + 1 - 0 = **8**.

2) $31^2 = 961$, and $32^2 = 1024$. So, there are 31 perfect squares between 1 and 1000, inclusive. Converting to a percent, 31 ÷ 1000 = 0.031, or 3.1% which, rounded to the nearest whole percent, is **3%**.

3) $\frac{4}{7} = 0.5714285714...$ A block of six digits (571428) repeats forever. To find the 127th digit, it is easiest to find the multiple of six which is closest to 127, namely 126. The 6th digit is 8, and every 6th digit after that is also an 8, including the 126th digit. Therefore, the 127th digit is **5**.

SOLUTIONS - Meet #4 - Category 2

ANSWERS

CATEGORY 2
GEOMETRY

1) 176.625

2) 1096.992

3) 11.25

1) If we know the circumference, then work backwards to find the length of the radius. Then find the area:

Circumference = (2) (π) (radius)
47.1 = (2) (3.14) (radius)
47.1 = (6.28) (radius)
radius = 47.1 ÷ 6.28
radius = 7.5

Area = (π) (radius)^2
= (3.14) (7.5)^2
= (3.14) (7.5) (7.5)
= **176.625** square meters.

2) Each straightaway is three times as long as the diameter of one semicircle. If the radius of one semicircle is 15, then its diameter is 30, and the straightaway is (3)(30), or 90. The two straightaways have a total length of 180.

Distance = (4) (perimeter)
= (4) (circumference of one circle+180)
= (4) [(2) (π) (radius) + 180]
= (4) [(2) (3.1416) (15) + 180]
= **1096.992** meters

3) The shaded area can be found by subtracting the sum of the areas of the three sectors from the area of the right triangle. The sector with center at point Y is a quarter of a circle, and the other two sectors are each an-eighth of a circle.

Shaded area = (triangle) - (sum of three sectors)
= (1/2) (B) (H) - ($2\pi R^2 \div 8 + \pi R^2 \div 4$)
= (1/2)(10)(10) - [2(3.1)(5^2)÷8 + (3.1)($5^2 \div 4$)]
= 50 - [2(3.1)(25)÷8 + (3.1)(25÷4)]
= 50 - [19.375 + 19.375]
= 50 - 38.75
= **11.25** square centimeters

SOLUTIONS - **Meet #4 - Category 3**

ANSWERS

CATEGORY 3
NUMBER THEORY

1) **Friday**

2) **2188**

3) **35,050**

1) This question is equivalent to, "What is one less than 312 mod 7?".
312 ÷ 7 = 44, with a remainder of 4. 312 days contains 44 seven-day weeks, with four days left over. Use the remainder to figure that one less than four days after Tuesday is **Friday**

2) This question is equivalent to, "What is the 365^{th} term in the following sequence: 4, 10, 16, 22, ..."
Each term is two less than a multiple of six, if compared to this sequence: 6, 12, 18, 24, ...
Two less than the 365^{th} multiple of six is

$$\begin{aligned} & (365)(6) - 2 \\ = \; & 2190 - 2 \\ = \; & 2188. \end{aligned}$$

3) First, find the value of the 100^{th} term of the sequence:
Each term is three less than a multiple of seven, if compared to this sequence: 7, 14, 21, ...
Three less than the hundredth multiple of seven is

$$\begin{aligned} & (100)(7) - 3 \\ = \; & 700 - 3 \\ = \; & 697. \end{aligned}$$

An easy way to find the sum of the first 100 terms is to add the first and last terms: 4 + 697 = 701.
Adding the second term to the next-to-last term yields 11 + 690 = 701. There are fifty such sums (half of 100 terms), so the sum of the first 100 terms is

$$\begin{aligned} & (50)\,(701) \\ = \; & \mathbf{35,050}. \end{aligned}$$

SOLUTIONS - Meet #4 - Category 4

ANSWERS

CATEGORY 4
ARITHMETIC

1) **46.40**
(The zero is optional)

2) **18**
(18.00 and 18.0 are also acceptable)

3) **23,982.77**

1) ($2/hour) (7 hours) + (4%) (18 pairs) ($45/pair)
= ($14) + (0.04) (18 pairs) ($45/pair)
= ($14) + ($32.40)
= $**46.40**.

2) The difference ($0.90) in the tip amount represents 20% - 15%, or 5% of the total bill.

5% (total bill) = $0.90
0.05 T = 0.90
T = 0.90 ÷ 0.05
T = 18

Therefore, the total bill was $**18**.

3) First year's interest: 16,990 (0.09) = 1529.10.
Second year's interest: (16,990 + 1529.10) (0.09)
= (18,519.1) (0.09)
= 1666.719
which rounds off to ≈ 1666.72
Third year's interest: (18,519.1 + 1666.72) (0.09)
= (20,185.82)(0.09)
= 1816.7238
which rounds off to ≈ 1816.72
Fourth year's interest: (20,185.82 + 1816.72)(0.09)
= (22,002.54)(0.09)
= 1980.2286
which rounds off to ≈ 1980.23

So, the total amount that Paul paid for the car, including interest, was

16,990 + 1529.10 + 1666.72 + 1816.72 + 1980.23
= $**23,982.77**.

SOLUTIONS - Meet #4 - Category 5

ANSWERS

CATEGORY 5
ALGEBRA

1) 23

2) 26

3) 11

1) A proportion can be used to solve this problem:

$$\frac{\text{Jennifer's height}}{\text{length of her shadow}} = \frac{\text{telephone pole's height}}{\text{length of its shadow}}$$

$$\frac{5}{7.2} = \frac{T}{33.12}$$

$$(7.2)\,T = (5)(33.12)$$
$$(7.2)\,T = 165.6$$
$$T = 165.6 \div 7.2$$
$$T = 23$$

Therefore, The height of the telephone pole is **23** ft.

2) Solve a system of equations in two variables:
Let T = the number of True-False questions
M = the number of multiple-choice questions

$$T + M = 43$$
$$6T + 5M = 232$$

Then multiply both members of the first equation by 5, so that subtracting the two equations will eliminate the variable M:

$$5(T + M) = 5(43)$$
$$6T + 5M = 232$$

Now distribute the 5 to simplify:

$$5T + 5M = 215$$
$$6T + 5M = 232$$

SOLUTIONS - Meet #4 - Category 5

Now subtract the top equation from the bottom equation:

$$\begin{aligned} 5T + 5M &= 215 \\ \underline{6T + 5M} &= \underline{232} \end{aligned}$$

$$T = 17$$

Now substitute 17 for T in either equation to find the value of M:

$$\begin{aligned} T + M &= 43 \\ 17 + M &= 43 \\ M &= 43 - 17 \\ M &= 26 \end{aligned}$$

Therefore, there are **26** multiple-choice questions.

3) For each proportion, the cross-products are equal, thus creating these two equations:

$5(N - 2) = 3(R - 1)$ and $3(N - 1) = 2R$

$5N - 10 = 3R - 3$ $\quad\quad$ $3N - 3 = 2R$

Rearranging each equation into standard form:

$5N - 3R = 7$ and $3N - 2R = 3$

$$\begin{aligned} 5N - 3R &= 7 \\ 3N - 2R &= 3 \end{aligned}$$

To eliminate the variable R, multiply both members of the top equation by 2, multiply both members of the bottom equation by -3, then add the two equations:

$$\begin{aligned} 2(5N - 3R) &= 2(7) \\ -3(3N - 2R) &= -3(3) \end{aligned}$$

Distribute to simplify:

$$\begin{aligned} 10N - 6R &= 14 \\ -9N + 6R &= -9 \end{aligned}$$

SOLUTIONS - Meet #4 - Category 5

Now add the two equations to eliminate the variable R:

$$\begin{array}{r} 10N - 6R = 14 \\ \underline{-9N + 6R = -9} \end{array}$$

$$N = 5$$

Substitute 5 for N in any equation to solve for R:

$$\begin{aligned} 3N - 2R &= 3 \\ 3(5) - 2R &= 3 \\ 15 - 2R &= 3 \\ 2R &= 15 - 3 \\ 2R &= 12 \\ R &= 12 \div 2 \\ R &= 6 \end{aligned}$$

Therefore, the value of $(N + R) = (5 + 6) = \mathbf{11}$.

SOLUTIONS - **Meet #4 - Category 6**

ANSWERS

CATEGORY 6
TEAM QUESTIONS

1) 64

2) 32

3) 9

4) 50

5) 97

6) 9

1) Let X = the degree measure of the smallest angle.
X + 4 = the degree measure of the next larger angle.
X + 8 = the degree measure of the largest angle.

Now use the concept that the sum of the measures of any triangle is 180 degrees:

$$\begin{aligned} X + (X+4) + (X+8) &= 180 \\ 3X + 12 &= 180 \\ 3X + 12 + (-12) &= 180 + (-12) \\ 3X &= 168 \\ 3X \div 3 &= 168 \div 3 \\ X &= 56 \end{aligned}$$

Therefore, the largest angle measures X + 8, or **64** degrees.

2) The calculator should allow students to explore a "trial & error" strategy. The two solutions are

$1^3 + 12^3 = 1729$, because $1 + 1728 = 1729$, and

$9^3 + 10^3 = 1729$, because $729 + 1000 = 1729$.

Therefore, M + A + T + H = 1 + 12 + 9 + 10 = **32**.

3) On the next page, there is a chart which tells the cent value for each letter of the alphabet.

SOLUTIONS - Meet #4 - Category 6

1	A
2	B
3	C
4	D
5	E
6	F
7	G
8	H
9	I
10	J
11	K
12	L
13	M
14	N
15	O
16	P
17	Q
18	R
19	S
20	T
21	U
22	V
23	W
24	X
25	Y
26	Z

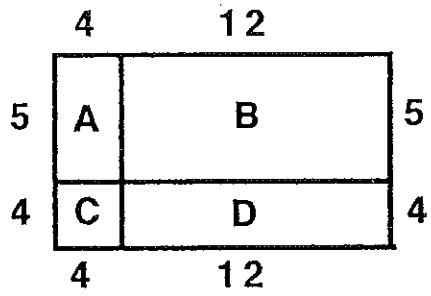
Here are the values for each day of the week:

Sunday = 19 + 21 + 14 + 4 + 1 + 25 = 84 cents
Monday = 13 + 15 + 14 + 4 + 1 + 25 = 72 cents
Tuesday = 20 + 21 + 5 + 19 + 4 + 1 + 25 = 95 cents
Wednesday = 23+5+4+14+5+19+4+1+25 = **$1.00**
Thursday = 20+8+21+18+19+4+1+25 = $1.16
Friday = 6 + 18 + 9 + 4 + 1 + 25 = 63 cents
Saturday = 19+1+20+21+18+4+1+25 = $1.09.

Therefore, it is Wednesday which is worth $1.00, and the word "Wednesday" contains 9 letters.

SOLUTIONS - Meet #4 - Category 6

4) The factors of the area numbers must be considered, as Area = Length x Width for all rectangles. A little trial and error will yield the following dimensions for each of the rectangles. The dimensions of rectangle A will become known automatically, once the dimensions of the other rectangles are known.



The perimeter of the original rectangle is

= $4 + 5 + 4 + 12 + 5 + 4 + 12 + 4$
= **50** yards.

5) The positive integers with an odd number of factors are the set of {perfect squares}. The perfect squares which fall into the range of 10 to 100, inclusive, are the following: 16, 25, 36, 49, 64, 81, and 100. The factors of those numbers are as follows:

16: 1, 2, 4, 8, 16
25: 1, 5, 25
36: 1, 2, 3, 4, 6, 9, 12, 18, 36
49: 1, 7, 49
64: 1, 2, 4, 8, 16, 32, 64
81: 1, 3, 9, 27, 81
100: 1, 2, 4, 5, 10, 20, 25, 50, 100

The only perfect squares with exactly five factors are 16 and 81. Their sum is 16+81 = **97**.

SOLUTIONS - Meet #4 - Category 6

6) $\sqrt[\frac{4\sqrt{B}}{2}]{B^2 - (E - A)^2 - D + C + 187}$

$= \sqrt[\frac{4\sqrt{32}}{2}]{32^2 - (97 - 64)^2 - 50 + 9 + 187}$

$= \sqrt[4\sqrt{16}]{1024 - (33)^2 - 50 + 9 + 187}$

$= \sqrt[2]{1024 - 1089 - 50 + 9 + 187}$

$= \sqrt{81}$

$= 9$