April 1998

INTERMEDIATE MATH LEAGUE OF EASTERN MASSACHUSETTS

MEET #5

CATEGORY 1 - MYSTERY
MEET #5 - APRIL, 1998

You may use a CALCULATOR Today!

① A palindrome is a numbers which reads the same as when its digits are reversed. For example, 27572 is a palindrome. What is the largest prime numbers less than 400 which is a palindrome?

② The product of three consecutive odd integers is 357,627. What is the sum of the smallest and largest of those three integers?

③ More than 20 students, but fewer than 40 students paid a total of $527 for concert tickets. If each ticket costs the same whole numbers of dollars, then how many students are in the group?

ANSWERS

① _____

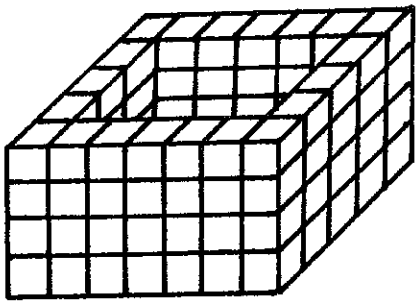
② _____

③ _____

CATEGORY 2 - GEOMETRY
Meet #5 - April, 1998

① The surface area of a cube is 168.54 square feet. How many cubic feet are in its volume?

② How many of the smallest cubes are there in this configuration, which was built entirely of sugar cubes? Assume that the hole goes all the way through from top to bottom.



③ A sphere has as many square inches in its surface area as it has cubic inches in its volume. How many inches long is the diameter of the sphere?

ANSWERS

① _____ cu. ft.

② _____ cubes

③ _____ inches

CATEGORY 3 - NUMBER THEORY
MEET #5 - April, 1998

① $M = \{2, 3, 5, 7, 11, 13\}$
$A = \{4, 7, 10, 13, 16\}$
$T = \{6, 12, 18, 24\}$
$H = \{3, 5, 7, 9\}$

List the elements in set P, arranged from smallest to largest, if $P = (M \cap H) \cup (A \cap T)$.

② Of the 126 8th graders at Weston Middle School,
108 are less than 6 feet tall,
47 have dark hair, and
34 are less than 6 feet tall and have dark hair.
How many are at least 6 feet tall and do not have dark hair?

③ All of the students who attend the Pickering Middle School in Lynn watched at least one event of the Winter Olympics. How many of the 423 Pickering students watched only one event? Use the data in the chart below:

Event	Number who watched
Bobsledding	196
Figure Skating	362
Ski Jumping	286
Bobsledding and Figure Skating	149
Figure Skating and Ski Jumping	241
Bobsledding and Ski Jumping	110
All three events	79

ANSWERS

① { _____ }

② _____

③ _____

CATEGORY 4 - ARITHMETIC

Meet #5 - April, 1998

① In a box is a collection of pennies, all the same size and same shape. There are 38 Lincoln-head pennies, 18 Indian-head pennies, and 28 Canadian pennies. Samantha will choose a penny, at random, from the box. What is the probability that she chooses an Indian-head penny? Express your answer as a fraction in lowest terms.

② $\{1, 2, 3, 4, 5, \ldots\}$ is the infinite set of natural numbers. An abundant number is a natural number, the sum of whose proper factors is greater than that natural number. For example, 36 is abundant, because $1+2+3+4+6+9+12+18 > 36$. What is the probability that a randomly selected natural number less than 31 is abundant? Express your answer as a fraction in lowest terms.

③ A vendor is selling balloons at the parade. She has 7 green, 4 yellow, and 5 red balloons left. Bob and Carol each wants a balloon, but does not care what color it is, so will choose one at random. What is the probability that Bob and Carol will choose a balloon of the same color? Express your answer as a fraction in lowest terms.

ANSWERS

① _____

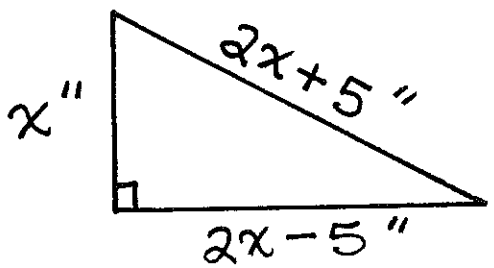
② _____

③ _____

CATEGORY 5 - ALGEBRA
Meet #5 - April, 1998

① There are <u>two</u> values of N which make the equation $x + 15 = 2x^2$ a true statement. Find the sum of those two values. Express your answers as a fraction in lowest terms.

②



Find the number of square inches in the area of this right triangle.

③ What positive number is 2.1 more than its reciprocal? Express your answer as a decimal.

ANSWERS

① _____

② _____ $in.^2$

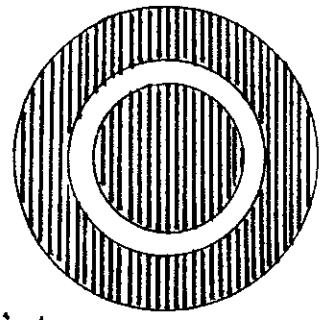
③ _____

CATEGORY 6 - TEAM QUESTIONS
MEET #5 - APRIL, 1998

① One end of a dog's 30-foot leash is attached to the corner of a rectangular house which measures 24 feet by 40 feet. The maximum area in which the dog can roam is $N \cdot \pi$ square feet. What is the value of N?

② Two different natural numbers are selected at random from 1 through 10, inclusive. The probability that their product is an odd number is $\frac{C}{D}$, which is in lowest terms. What is the value of $C+D$?

③ A target consists of three concentric circles (they have the same center), having radii of 4, 6, and 12 cm, respectively. What is the probability that a dart which hits the target will hit a shaded region? Express your answer as a fraction in lowest terms. The figure is not necessarily drawn to scale.



④ What is the remainder when 5^{27} is divided by 7?

⑤ The outer surface of a spherical gas tank must be painted. The diameter of the sphere is 30 yards. One gallon of paint will cover 125 square feet. What is the least number of gallons of paint which must be purchased in order for the tank to be painted? Use $\pi \approx 3.142$.

⑥ Evaluate the following expression, using the answers to #1-5 to replace the letters W, Y, E, H, and R, respectively:

$$\sqrt[4]{(4)(Y-1)\left(\frac{EWR}{W-R}\right) - (2H+1)}$$

ANSWERS

① _____ = W
② _____ = Y
③ _____ = E
④ _____ = H
⑤ _____ = R
⑥ _____

SOLUTIONS - Meet #5 (page 1)

CATEGORY 1

① 383

② 142

③ 31

① 393? No (divisible by 3)
383? Yes, it's prime!

② The middle # should be roughly the cube root of 357,627, which is about 71.

$69 \times 71 \times 73 = 357,627$

∴ The SUM of the smallest and largest is $69 + 73$, or 142.

③ 527 must be the product of two prime numbers, since 527 is odd. Try dividing 527 by different odd numbers between 20 and 40 until the quotient is a whole number. $527 \div 31 = 17$, so there are 31 students in the group. (Note: Each ticket costs #17 - a whole # of dollars)

CATEGORY 2

① 148.877

② 80

③ 6

① Let x = the length of one side of the cube.
x^2 = the area of one surface
$6x^2$ = the total surface area

$6x^2 = 168.54$
$x^2 = \frac{168.54}{6}$
$x^2 = 28.09$
$x = \sqrt{28.09}$

→ $x = 5.3$

The volume $= x^3$
$= 5.3^3$
$= 148.877$ cu. ft.

② Front: $7 \times 4 = 28$
Back: $7 \times 4 = 28$
Left side between front and back: $4 \times 3 = 12$
Right side between front and back: $4 \times 3 = 12$
Total: $28 + 28 + 12 + 12 = 80$

③ $4\pi r^2 = \frac{4}{3}\pi r^3$ Surface area = volume.
$12\pi r^2 = 4\pi r^3$ mult. both sides by 3.
$3r^2 = r^3$ Divide both sides by 4π.
$3 = r$ Divide both sides by r^2.
If the radius is 3, then the diameter is 6.

SOLUTIONS – Meet #5 (page 2)

CATEGORY 3

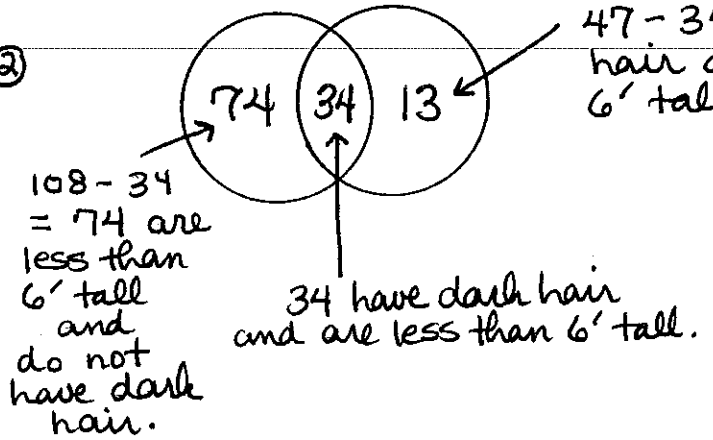
① {3,5,7}

② 5

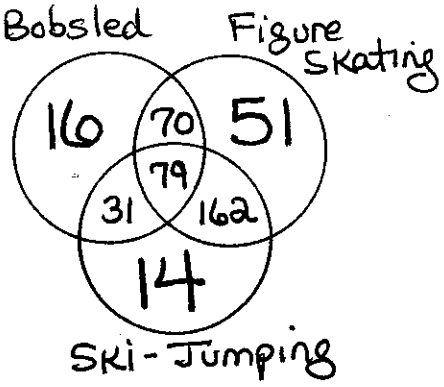
③ 81

① $P = (M \cap H) \cup (A \cap T)$
$P = \{3,5,7\} \cup \{\ \}$
$P = \{3,5,7\}$

← Order is <u>important</u> – smallest to largest!

② 

The number who are at least 6′ tall <u>and</u> do not have dark hair is $126 - (74 + 34 + 13)$
$= 126 - 121$
$= \boxed{5}$

③ 

The numbers written in large numerals represent the students who watched only one event.

$16 + 14 + 51 = \boxed{81}$

Solutions – Meet #5 (page 3)

Category 4

① $\frac{3}{14}$

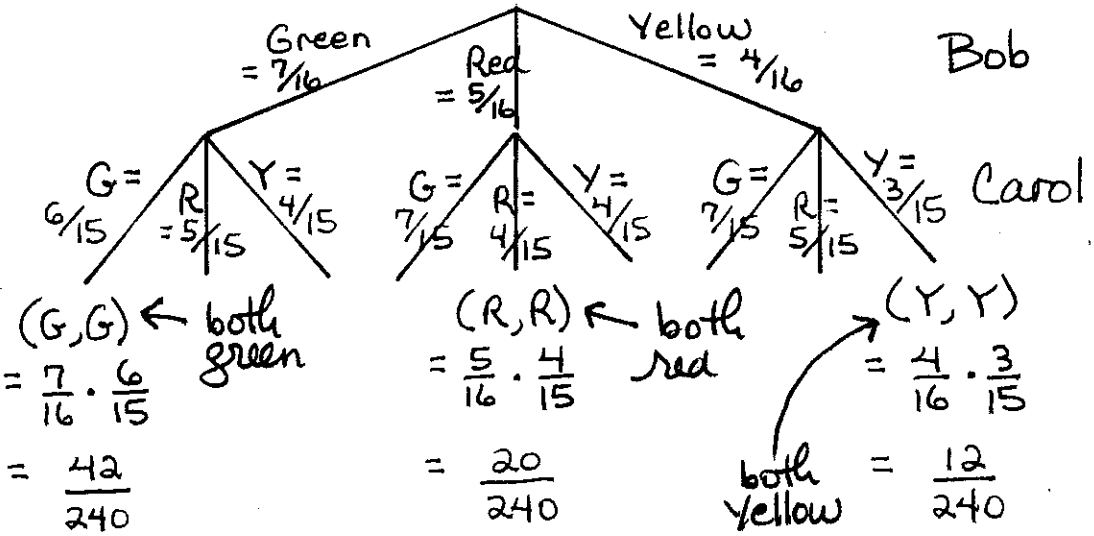
② $\frac{1}{6}$

③ $\frac{37}{120}$

① $\frac{\#\text{ of Indian-head pennies}}{\text{Total \# of pennies}} = \frac{18}{84} = \frac{3}{14}$ (lowest terms)

② The only abundant numbers which are less than 31 are 12, 18, 20, and 24, and 30 (Five in all) $\frac{5}{30} = \frac{1}{6}$

③ Since Bob and Carol can't pick the same balloon, this problem is equivalent to a "Grecian urn without replacement" problem. This tree diagram may help:



(G,G) ← both green $= \frac{7}{16} \cdot \frac{6}{15} = \frac{42}{240}$

(R,R) ← both red $= \frac{5}{16} \cdot \frac{4}{15} = \frac{20}{240}$

(Y,Y) ← both Yellow $= \frac{4}{16} \cdot \frac{3}{15} = \frac{12}{240}$

$\frac{42}{240} + \frac{20}{240} + \frac{12}{240} = \frac{74}{240}$, or $\frac{37}{120}$

Category 5

① $\frac{1}{2}$ only!

② 1500

③ 2.5 only!

① $x + 15 = 2x^2$ original equation

$2x^2 - x - 15 = 0$ transformed so one member is 0.

$(2x+5)(x-3) = 0$ Factored

$2x + 5 = 0$ or $x - 3 = 0$ Zero factor theorem

$2x = -5$ $x = 3$

$x = -5/2$

$x = -2.5$

The sum is $-2.5 + 3$

$= 0.5$

$= \frac{1}{2}$ (The answer must be a fraction!)

SOLUTIONS - Meet #5 (page 4)

② Use the Pythagorean Theorem to find the length of the base and altitude:

$$\text{leg}^2 + \text{other leg}^2 = \text{hypotenuse}^2$$
$$x^2 + (2x-5)^2 = (2x+5)^2$$
$$x^2 + 4x^2 - 20x + 25 = 4x^2 + 20x + 25$$
$$5x^2 - 20x + 25 = 4x^2 + 20x + 25$$
$$x^2 - 40x = 0$$
$$x(x-40) = 0$$
$$x = 0 \quad \text{or} \quad x - 40 = 0$$
$$x = 40 \text{ (altitude)}$$
$$2x - 5 = 75 \text{ (base)}$$

(discard, as no length of a triangle can be 0.)

$\therefore$ The area $= \frac{1}{2}(\text{base})(\text{altitude})$
$= \frac{1}{2}(75)(40)$
$= 1500 \text{ sq. in.}$

③ Let x = the number
$\frac{1}{x}$ = the number's reciprocal

$x = 2.1 + \frac{1}{x}$ translated

$x^2 = 2.1x + 1$ mult. both sides by x.

$x^2 - 2.1x - 1 = 0$ transformed

$10x^2 - 21x - 10 = 0$ mult. both sides by 10.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{21 \pm \sqrt{(-21)^2 - 4(10)(-10)}}{2(10)}$$
$$= \frac{21 \pm \sqrt{441 + 400}}{20}$$
$$= \frac{21 \pm \sqrt{841}}{20}$$
$$= \frac{21 \pm 29}{20}$$
$$= \frac{50}{20} \text{ or } \frac{-8}{20}$$
$$= 2.5 \text{ or } -0.4$$

Accept the positive value. Reject the negative value.

SOLUTIONS - MEET #5 (page 5)

CATEGORY 6

① 684

② 11

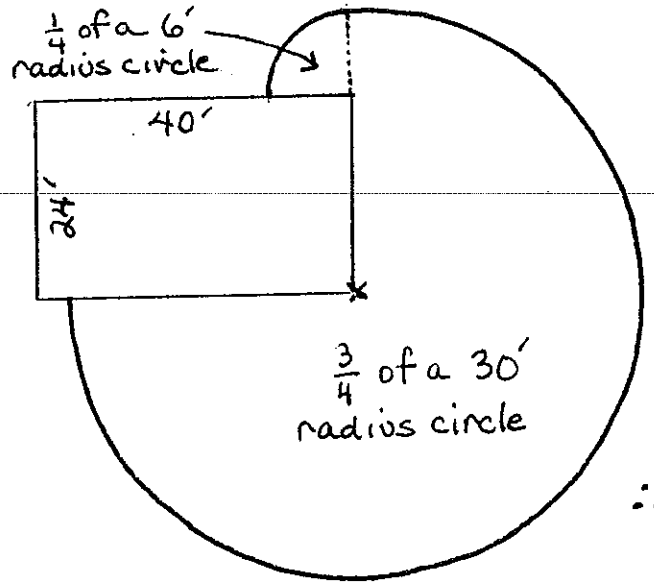
③ $\frac{31}{36}$

④ 6

⑤ 204

⑥ 10

① It should not matter from which corner the leash is attached. I will choose one randomly:



$$\text{Area} = \frac{3}{4}\pi r_1^2 + \frac{1}{4}\pi r_2^2$$

where $r_1 = 30$, $r_2 = 6$

$$= \frac{3}{4}\pi(30^2) + \frac{1}{4}\pi(6^2)$$
$$= \frac{3}{4}\pi(900) + \frac{1}{4}\pi(36)$$
$$= 675\pi + 9\pi$$
$$= 684\pi$$

$\therefore$ N = 684

② The chart below shows all possible products of two different natural numbers chosen from 1–10 at random:

	1	2	3	4	5	6	7	8	9	10
1	X	2	(3)	4	(5)	6	(7)	8	(9)	10
2	2	X	6	8	10	12	14	16	18	20
3	(3)	6	X	12	(15)	18	(21)	24	(27)	30
4	4	8	12	X	20	24	28	32	36	40
5	(5)	10	(15)	20	X	30	(35)	40	(45)	50
6	6	12	18	24	30	X	42	48	54	60
7	(7)	14	(21)	28	(35)	42	X	56	(63)	70
8	8	16	24	32	40	48	56	X	72	80
9	(9)	18	(27)	36	(45)	54	(63)	72	X	90
10	10	20	30	40	50	60	70	80	90	X

The odd products are circled.

$$\frac{\#\text{ odd products}}{\#\text{ total products}} = \frac{20}{90}$$
$$= \frac{2}{9}$$
$$= \frac{C}{D}$$

$\therefore$ C + D = 2 + 9 = (11)

③ Shaded area = smallest circle + outer "ring"
= smallest circle + (largest circle − medium circle)

$$= \pi \cdot 4^2 + (\pi \cdot 12^2 - \pi \cdot 6^2)$$
$$= 16\pi + (144\pi - 36\pi)$$
$$= 16\pi + 108\pi$$
$$= 124\pi$$

(next page)

3) continued ...

The probability that a dart which hits the target will hit the shaded area is

$$\frac{\text{area of shaded region}}{\text{area of the target}} = \frac{124\pi}{144\pi} = \frac{124}{144} = \frac{62}{72} = \boxed{\frac{31}{36}}$$

④

Power of 5	Remainder when ÷7
5^1	5
5^2	4
5^3	6
5^4	2
5^5	3
5^6	1
5^7	5
5^8	4
5^9	6
⋮	⋮

This block of six remainders repeats in this sequence forever.

For 5^{27}, 27 is three more than a multiple of 6.

For $5^6, 5^{12}, 5^{18}, 5^{24}, \ldots$ to be divided by 7, the remainder would be 1.

For 5^{27}, count three ahead, and the remainder would be 6.

⑤ The surface area of a sphere is $4\pi r^2$, where r is the radius of the sphere. Diameter is 30, so $r = 15$.

$$SA = 4\pi r^2$$
$$\approx 4(3.142)(15^2)$$
$$\approx 12.568(225)$$
$$\approx 2827.8 \text{ square yards.}$$

Since the paint covers 125 square feet per gallon, convert square yards to square feet (9 sq. ft $= 1$ sq. yd):

$2827.8(9) = 25450.2$ square feet.

Now divide by 125 to find the number of gallons of paint:

$25{,}450.2 \div 125 \approx 203.6$ gallons.

∴ 204 gallons must be purchased.

⑥ $\sqrt[4]{(4)(Y-1)\left(\frac{EWR}{W-R}\right) - (2H+1)}$

$= \sqrt[4]{(4)(11-1)\left(\frac{\frac{31}{36}\cdot 684 \cdot 204}{684-204}\right) - (2\cdot 6+1)}$

$= \sqrt[4]{(4)(10)\left(\frac{\frac{31}{\cancel{36}}\cdot \frac{19}{\cancel{684}}\cdot 204}{480}\right) - (12+1)}$

$= \sqrt[4]{\cancel{(4)}^{1}\cancel{(10)}\left(\frac{120,156}{\cancel{480}_{12}}\right) - 13}$

$= \sqrt[4]{10,013 - 13}$

$= \sqrt[4]{10,000}$

$= 10$