1998

INTERMEDIATE MATH LEAGUE OF EASTERN MASSACHUSETTS

MEET #4

CATEGORY 1 - MYSTERY
MEET #4

CALCULATORS may be used for all categories today !!

① The sum of the squares of two consecutive odd integers is 1570. What is the larger of those two integers?

② Waldo had to write numbers on the tickets for a school basketball game. The ticket numbers were consecutive integers from 1 to 275, inclusive (including 1 and 275). How many digits did Waldo write in all?

③ If H and A are whole numbers, and each one is less than 7, and $H^A = 4096$, then find the value of H.

ANSWERS

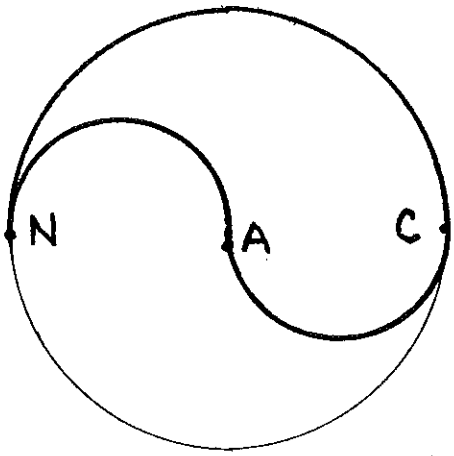
① _____

② _____

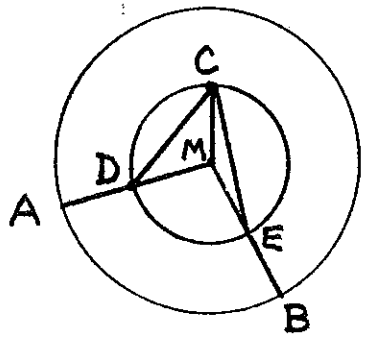
③ _____

CATEGORY 2 - GEOMETRY 1998
MEET #4

① $\overline{NC}$ is a diameter of circle A. AC = 35 inches. Find the perimeter of the shape which is highlighted with the thick path. Use $\pi \approx 3.14$. Round your final answer to the nearest tenth of an inch.

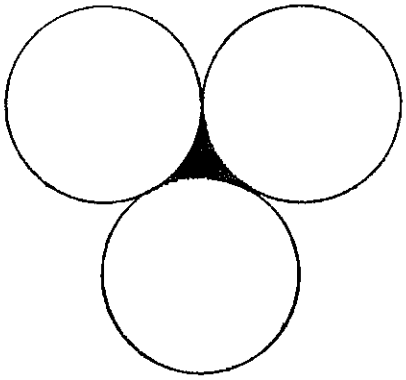


②



Point M is the center of the two circles shown at the left. The measure of minor arc $\overset{\frown}{AB} = 108°$. $\overset{\frown}{DC} = \overset{\frown}{CE}$. D is on $\overline{AM}$; E is on $\overline{MB}$. Find the measure of angle ADC if it is less than 180°.

③ Three congruent circles are externally tangent, with any two intersecting each other at one point. If the radius of each circle is 30 centimeters, then find the area of the shaded region, to the nearest tenth of a square centimeter. Use $\pi \approx 3.1416$.



ANSWERS

① _____ inches

② _____ degrees

③ _____ sq. cm

CATEGORY 3 - NUMBER THEORY
MEET #4 1998

① Find the value of □ in the following sequence:

7 11 19 31 47 □

② Jethro has been having a hiccup attack every 17 minutes. His first attack was at 10:42 P.M. In all, he had 132 hiccup attacks. At what time did he have his final attack? You must include either A.M. or P.M. in your answer.

③ Erin likes to count the stars in the night sky. She counted 23 on the first night, 31 on the second night, 39 on the third, and 47 on the fourth, and so on. Assuming that the night sky is clear enough each night to count stars, how many stars did Erin count in a year, if a year has 365 days (and nights!)?

ANSWERS

① _____

② _____

③ _____

CATEGORY 4 - ARITHMETIC
MEET #4

① Ben earned a score of 85% correct on a test which had 80 questions. How many of the questions did Ben answer incorrectly?

② Tory received a 25% discount, and only had to pay $28.68 for a desk lamp. What was the original cost of the lamp?

③ Moe sold a model T Ford to Joe, and made a profit of 20%. Joe sold it to Larry, and made a 10% profit. Larry sold it to Curly, and took a 5% loss. Moe had originally purchased it from Shemp for $960. How many dollars did Curly pay for the model T Ford?

ANSWERS

① _____

② $_____

③ $_____

CATEGORY 5 – ALGEBRA
MEET # 4

① John noticed that he and three of his friends had birthdays on consecutive days in March. The sum of the dates is 70. John's birthday was later in March than was any of his friends'. On what day in March was John born?

② At a speed of 55 miles per hour, how many minutes would it take to travel 32 miles? Round your answer to the nearest whole number of minutes.

③ The girls went to lunch at Montezuma's on Monday and ordered the following:

Name	# of tacos	# of drinks
Jennifer	2	2
Wanda	3	1

They paid a total of $6.17.
The boys went out to lunch at Montezuma's on Tuesday, spent a total of $8.47, and ordered:

Name	# of tacos	# of drinks
Mark	1	1
David	3	1
Jon	3	2

ANSWERS

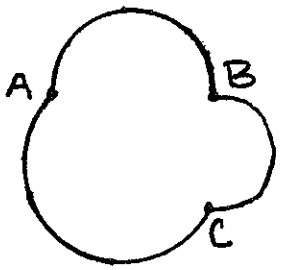
① March ____

② _____ minutes

③ $ _____

The whole group of boys and girls went to lunch at Montezuma's on Wednesday and ordered a total of 10 tacos and 8 drinks. What was the total cost of Wednesday's lunch?

CATEGORY 6 - TEAM QUESTIONS 1998
MEET #4

① 

$\overline{AB}$, $\overline{BC}$, and $\overline{AC}$ are the diameters of three semicircles. $\overline{AB} \perp \overline{BC}$. The radius of the largest semicircle is 39 inches, and the radius of the smallest semicircle is 15 inches. If the area of the enclosed region is $W\pi + N$ square inches, then find the value of $W + N$.

② What number should be added to both the numerator and denominator of $\frac{5}{8}$ to create a fraction whose value is $\frac{15}{16}$?

③ The 28th term of an arithmetic sequence is 335, and the 43rd term is 515. What is the value of the 100th term?

④ What is the value of N, if the difference between 40% of N and $\frac{7}{8}$ of N is 266? N is positive.

⑤ Solve for W: $\dfrac{2\frac{3}{7}}{9} = \dfrac{34}{W}$

⑥ Let the answers to questions #1-5 be represented by the letters A, B, C, D, and E, respectively. Evaluate the following expression:

$$\sqrt{\frac{C - \left(\sqrt{\frac{AD}{EB}}\right)}{6}} + \frac{80E}{9B}$$

ANSWERS

① _____ = A
② _____ = B
③ _____ = C
④ _____ = D
⑤ _____ = E
⑥

1998

SOLUTIONS - Meet #4

CATEGORY 1

① 29

② **717**

③ 4

① A good estimate for one of the integers would be the square root of half of 1570:
$1570 \approx 1600$, $\frac{1}{2}$ of $1600 = 800$, $\sqrt{800} \approx 28$.
Try 27 and 29: $27^2 + 29^2 = 729 + 841 = 1570$. Voila! The larger # is 29.

②

	numbers with 1 digit	numbers with 2 digits	numbers with 3 digits
	1 – 9	10 – 99	100 – 275
# of numbers:	9	90	176
total # of digits:	9	180	**528**

$9 + 180 + \mathbf{528} = \mathbf{717}$

③ H must be an even number. One could try raising the numbers 2, 4, or 6 to various powers.
OR, work backwards by finding the prime factorization of 4096:
$4096 = 2^{12}$, or 4^6, or 8^4, or 16^3. The only expression where both the base and exponent are less than 7 is 4^6. $\therefore H = 4$.

CATEGORY 2

① 219.8

② 153

③ 145.1

① Perimeter = $\overset{\frown}{NC} + \overset{\frown}{NA} + \overset{\frown}{AC}$ IF $\overset{\frown}{AC} = 35$ $\overset{\frown}{AC} = \frac{70}{11}$

IF $\overset{\frown}{AC} = 35$ $= \frac{1}{2}\pi d_1 + \frac{1}{2}\pi d_2 + \frac{1}{2}\pi d_3$

$\approx \frac{1}{2}(3.14)(70) + \frac{1}{2}(3.14)(35) + \frac{1}{2}(3.14)(35)$

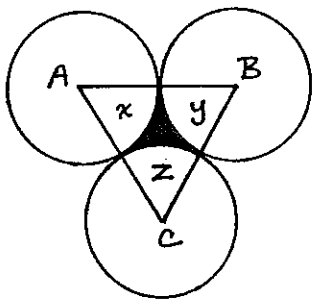
$\approx 109.9 + 54.95 + 54.95$

≈ 219.8 in.^{2}

(This is also the circumference of circle A!)

② There is a long series of connections: $\angle ADC$ is the supplement of $\angle CDM$, which can be found if the other two interior angles of $\triangle CDM$ are known. Since $m\overset{\frown}{AB} = 108°$, then the measure of central angle $DME = 108°$. Since $\overset{\frown}{DC} = \overset{\frown}{CE}$, then central angle $DMC = \angle CME = \frac{1}{2}(360 - 108) = 126°$, so central angle $DMC = 126°$. $\triangle DMC$ is isosceles, as radius $\overline{MD} = \overline{MC}$, so the base angles of $\triangle DMC$ are equal. $m\angle CDM = \frac{1}{2}(180 - 126) = 27°$.
$\therefore m\angle ADC = 180 - 27 = 153°$.

③ Hint: draw radii to create an equilateral triangle, as shown, $\triangle ABC$. The shaded area can be found by subtracting the combined area of the three sectors, x, y, and z, from the area of $\triangle ABC$.



$x+y+z$ = area of a semicircle
$= \frac{1}{2}\pi r^2$
$\approx \frac{1}{2}(3.1416)(30^2)$
$\approx \frac{1}{2}(3.1416)(900)$
≈ 1413.72

area of $\triangle ABC = \frac{1}{2}(\text{base})(\text{height})$
$\approx \frac{1}{2}(60)(51.9615)$
≈ 1558.845

$\therefore \triangle ABC - (x+y+z)$
$\approx 1558.845 - 1413.72$
$\approx 145.125 \approx 145.1$ (nearest tenth!)

The height can be found by using the Pythagorean Theorem:
$a^2+b^2=c^2$
$30^2+b^2=60^2$
$900+b^2=3600$
$b^2=2700$
$b=\sqrt{2700}$
$b \approx 51.9615$

CATEGORY 3

① 67

② 11:49 A.M. (must have A.M.)

③ 539,835

① 7, 11, 19, 31, 47, 67

first sequ. of differences: +4, +8, +12, +16, +20

second sequ. of differences: +4, +4, +4, +4

② The final attack occurred 131(17) minutes, or 2227 minutes, after the first one (NOT 132(17)!).
2227 minutes = 37 hours, 7 minutes.
10:42 P.M. + 37 hours, 7 minutes = 11:49 A.M.

③ This question is equivalent to "find the sum of the first 365 terms of the following arithmetic sequence: 23, 31, 39, 47, ...", where each term is 8 more than the one preceding it.
The 365th term is $365(8)+15$, since each term is 15 more than a multiple of 8.
$365(8)+15 = 2935$.
The sum of the 365 terms is:
$\frac{365}{2}(23+2935) = 539,835.$

SOLUTIONS - Meet #4 (page 3)

CATEGORY 4

① 12
② 38.24
③ 1203.84

① # of incorrect answers = 15% of 80
$= .15 \cdot 80$
$= 12$

② Let x = the original cost of the desk lamp

$x - 25\%\text{ of } x = 28.68$
$75\%\text{ of } x = 28.68$
$.75x = 28.68$
$x = \frac{28.68}{.75}$
$x = 38.24$

③ Joe bought the car for $960 + 20\%$ of 960
$= 960 + .2(960)$
$= 960 + 192$
$= \$1152.$

Larry bought it for $1152 + 10\%$ of 1152
$= 1152 + 115.20$
$= \$1267.20$

Curly bought it for $1267.20 - 5\%$ of 1267.20
$= 1267.20 - .05(1267.20)$
$= 1267.20 - 63.36$
$= \$1203.84$

CATEGORY 5

① 19
② 35
③ 14.02

① x, $x+1$, $x+2$, $x+3$ } The dates of the four birthdays

$x + x+1 + x+2 + x+3 = 70$
$4x + 6 = 70$
$4x = 64$
$x = 16$
$x+3 = 19$ (John's birthday was last.)

② $\frac{55 \text{ miles}}{60 \text{ minutes}} = \frac{32 \text{ miles}}{x \text{ minutes}}$

$55x = 32(60)$
$55x = 1920$
$x = \frac{1920}{55}$
$x \approx 34.909\ldots$
$x \approx 35$ (nearest whole #)

③ The girls ordered a total of 5 tacos and 3 drinks:

$$5T + 3D = 6.17$$

The boys ordered a total of 7 tacos and 4 drinks:

$$7T + 4D = 8.47$$

Solve the system of two equations to find the cost of 1 taco and 1 drink, then answer the question posed:

$$\left.\begin{aligned} 5T + 3D &= 6.17 \\ 7T + 4D &= 8.47 \end{aligned}\right\} \Rightarrow \begin{aligned} 35T + 21D &= 43.19 \\ 35T + 20D &= 42.35 \\ \hline 1D &= 0.84 \end{aligned}$$

∴ A drink costs 84¢.

$$\begin{aligned} 5T + 3D &= 6.17 \\ 5T + 3(0.84) &= 6.17 \\ 5T + 2.52 &= 6.17 \\ 5T &= 3.65 \\ T &= \frac{3.65}{5} \\ T &= 0.73 \end{aligned}$$

∴ A taco costs 73¢.

10 tacos and 8 drinks cost:

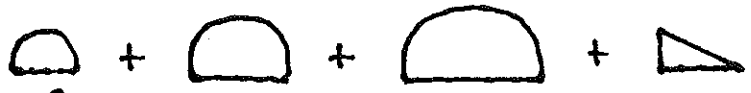
$$10(0.73) + 8(0.84)$$
$$= 7.30 + 6.72$$
$$= \$14.02$$

CATEGORY 6

① 2601
② 40
③ 1199
④ 560
⑤ 126
⑥ 15

① The area includes three semicircles and a right triangle. The diameter, $\overline{AC}$, of the largest semicircle is 78" because the radius is 39". The diameter of the smallest semicircle is 30". The diameter of the remaining semicircle can be found using the Pythagorean Theorem:

$$a^2 + b^2 = c^2$$
$$30^2 + b^2 = 78^2$$
$$900 + b^2 = 6084$$
$$b^2 = 6084 - 900$$
$$b^2 = 5184$$
$$b = \sqrt{5184}$$
$$\therefore b = 72$$

Area =  + + +

$= \frac{\pi r^2}{2} + \frac{\pi r^2}{2} + \frac{\pi r^2}{2} + \frac{bh}{2}$

$= \frac{\pi(15)^2}{2} + \frac{\pi(36)^2}{2} + \frac{\pi(39)^2}{2} + \frac{(72)(30)}{2}$

$= 112.5\pi + 648\pi + 760.5\pi + 1080$

$= \underbrace{1521\pi}_{\downarrow W} + \underbrace{1080}_{\downarrow N}$

$\therefore W + N = 1521 + 1080 = 2601$

② $\frac{5+N}{8+N} = \frac{15}{16}$

$15(8+N) = 16(5+N)$

$120 + 15N = 80 + 16N$

$40 = N$

③ The constant difference between any two consecutive terms can be found as follows:

1. Find the difference $515 - 335$, $= 180$
2. Find the number of terms from the 28th to the 43rd, which is $43 - 28$, or 15
3. Divide 180 by 15, $= 12$.

$\therefore$ The value of the 100th term is equal to the value of the 43rd term, plus 57 increments of 12:

$515 + 57(12)$

$= 515 + 684$

$= 1199$

Solutions - Meet #4 (page 6)

④ $\frac{7}{8}N - 40\%\,N = 266$

$\frac{7}{8}N - \frac{2}{5}N = 266$

$\frac{35}{40}N - \frac{16}{40}N = 266$

$\frac{19}{40}N = 266$

$\frac{40}{19} \cdot \frac{19}{40}N = \frac{40}{19} \cdot 266$

$N = 560$

⑤ $\frac{2\frac{3}{7}}{9} = \frac{34}{W}$

$2\frac{3}{7}W = 34(9)$

$2\frac{3}{7}W = 306$

$W = \frac{306}{2\frac{3}{7}}$

$W = 126$

⑥ $\sqrt{\frac{C - \left(\sqrt{\frac{AD}{EB}}\right)}{6} + \frac{80E}{9B}}$

$= \sqrt{\frac{1199 - \left(\sqrt{\frac{2601(560)}{126(40)}}\right)}{6} + \frac{80(126)}{9(40)}}$

$= \sqrt{\frac{1199 - \left(\sqrt{\frac{1456560}{5040}}\right)}{6} + 28}$

$= \sqrt{\frac{1199 - \sqrt{289}}{6} + 28}$

$= \sqrt{\frac{1199 - 17}{6} + 28}$

$= \sqrt{\frac{1182}{6} + 28}$

$= \sqrt{197 + 28}$

$= \sqrt{225}$

$= 15$