Jan 98

Jan 98??

INTERMEDIATE MATH LEAGUE OF EASTERN MASSACHUSETTS

MEET #3

CATEGORY 1 - MYSTERY Jan 1998
MEET #3

① N and M are positive whole numbers, and $N+M=16$. Find the positive difference between the greatest possible product of $N \cdot M$ and the lowest possible product of $N \cdot M$.

② Joe ate 145 peanuts in 5 days. Each day, he ate 6 more than on the day before. How many peanuts did Joe eat on the 4th day?

③ Julie spent 3/4 of her savings account to buy a parrot, and 3/4 of the remainder on supplies for the parrot. She had \$23 left. How many dollars did the parrot cost?

ANSWERS

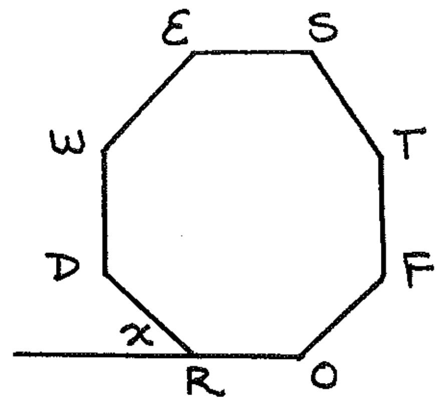
① ______

② ______

③ \$ ______

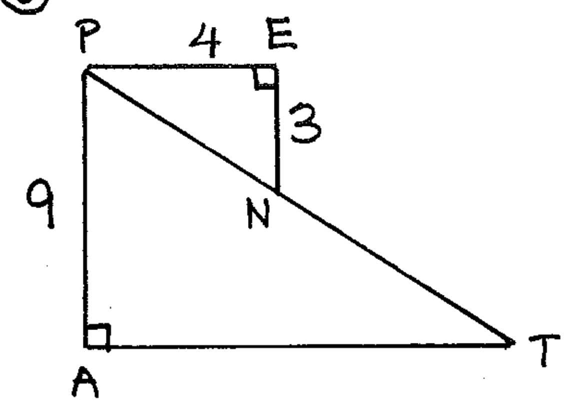
CATEGORY 2 - GEOMETRY Jan 1998
MEET # 3

① WESTFORD is a regular octagon. How many degrees are in the exterior angle marked x ?



② Bernard travelled 18 Km east, then 10 Km north, then 6 Km east. How many Km is Bernard from his starting point ?

③



If $\overline{PE} \parallel \overline{AT}$, then find how many units are in the perimeter of polygon PENTA

ANSWERS

① ______ degrees

② ______ Km

③ ______ units

CATEGORY 3 - NUMBER THEORY
Meet #3

① 243 is a numeral written in base 5. What is its value in base 10? In other words,

$$243_{\text{base }5} = \_\_\_\_\_\__{\text{base }10}$$

② Kermit has a collection of ancient coins, as follows:

5 Harpos, each worth 1 cent
3 Zeppos, each worth 7 cents
2 Grouchos, each worth 49 cents.

Fozzie has a collection of coins which is worth the same as Kermit's collection. Fozzie has:

Some Rickis, each worth 81 cents
Some Lucys, each worth 27 cents
Some Desis, each worth 9 cents
Some Ethels, each worth 3 cents
Some Freds, each worth 1 cent.

Fozzie has no more than two of any one type of coin.
How many coins does Fozzie have in all?

③ Evaluate. Write your final answer in scientific notation:

$$\frac{(3.2 \times 10^{7}) \cdot (4.2 \times 10^{-2})}{(800 \times 10^{-5}) \cdot (0.014 \times 10^{11})}$$

ANSWERS

① ______

② ______

③ ______

CATEGORY 4 - ARITHMETIC 1998
MEET #3

① Evaluate. Express your answer as a mixed numeral in simplest form:

$$2^3 + 2^2 + 2^1 + 2^0 + 2^{-1} + 2^{-2} + 2^{-3}$$

② Evaluate: $(8^3) \cdot (4^{-5}) \cdot (2^{-6}) \cdot (16^2)$

③ Evaluate:

$$\left(\sqrt[3]{\left(\sqrt[6]{\left(\left(\sqrt[5]{32}\right)^3\right)^{-2}}\right)^6}\right)^{-1}$$

ANSWERS

① ______

② ______

③ ______

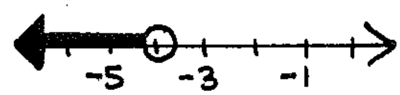
CATEGORY 5 - Algebra
MEET #3

① Evaluate: $|-6| + 3|7| - 4|-2| + 5|0|$

② If x is an integer, then for how many values of x is $\dfrac{6}{|x+2|}$ a positive integer? Don't list them - just tell how many there are.

③ For what value of W is the solution set of

$$4N - 2(3N - 5) + 5W > 33$$

given by the following graph:  ?

ANSWERS

① ______

② ______

③ ______

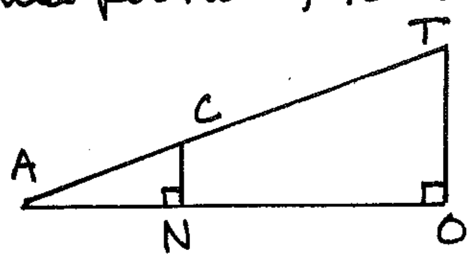
CATEGORY 6 - TEAM QUESTIONS JAN '98
MEET # 3

① How many degrees are in one interior angle of a regular pentadecagon, which is a 15-sided polygon?

② If $3^x = 12$, then what is the value of 3^{x+2}?

③ The average of two numbers, H and P, is 27, while the product of H and P is 713. What is the positive difference between H and P?

④ Jim lives in the town of Lynn. He drove to his friend's house in Swampscott, averaging 40 miles per hour, then drove back home to Lynn along the same route, averaging 60 miles per hour. What was Jim's average rate of speed, in miles per hour, for the entire round trip?

⑤ 

$\overline{CN}$ is parallel to $\overline{TO}$.
$\overline{AC}$ is ½ of $\overline{CT}$.
CN = 7 inches.
AN = 24 inches.
How many inches are in the perimeter of quadrilateral NCTO?

⑥ If A, B, C, D, and E represent the answers to questions # 1-5, respectively, then evaluate:

$$\frac{\frac{D}{C}}{E-(B-C)} \cdot C \cdot \frac{A}{B} \cdot \frac{E}{D}$$

ANSWERS

① ________ = A
② ________ = B
③ ________ = C
④ ________ = D
⑤ ________ = E

SOLUTIONS - Meet #3

CATEGORY 1

① 49
② 35
③ 276

① Greatest product $= 8 \cdot 8 = 64$
Least product $= 1 \cdot 15 = 15$
Difference $= 64 - 15 = 49$.

② Trial + error: $145 \div 5$, or 29, should be close to how many peanuts Joe ate on the middle (3rd) day. Check: $17+23+29+35+41 = 145$. Yes! ∴ Joe ate 35 peanuts on the 4th day.

③ Number-sense reasoning, using a geometric model:

X	X	X	#23
X	X	X	O
X	X	X	O
X	X	X	O

$\frac{3}{4}$ of savings — remainder

The 16 squares represent # in Julie's savings account. The 12 Xs represent the 3/4 of her savings account which she spent to buy the parrot. 3/4 of the remainder is represented by the 3 Os. #23 is left, which is 1/16 of what was originally in Julie's savings.

∴ Multiply 23 by 12 (the # of Xs) = #276.

Algebraic reasoning:

Let x = # in savings account

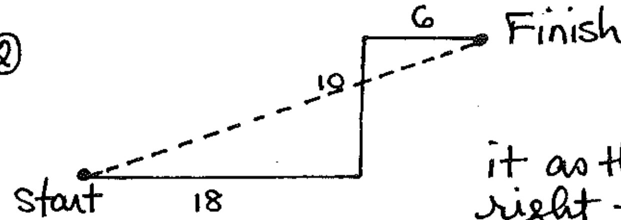
$$\text{then} \quad x - \tfrac{3}{4}x - \tfrac{3}{4}\left(x - \tfrac{3}{4}x\right) = 23$$
$$x - \tfrac{3}{4}x - \tfrac{3}{4}x + \tfrac{9}{16}x = 23$$
$$\tfrac{1}{16}x = 23$$
$$16 \cdot \tfrac{1}{16}x = 16 \cdot 23$$
$$(\text{\# originally in savings}) \quad x = 368$$
$$(\text{cost of parrot}) \quad \tfrac{3}{4}x = 276$$

CATEGORY 2

① 45
② 26
③ 38

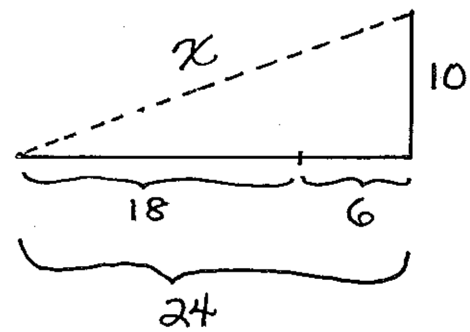
① Quickest method: The sum of the exterior angles of any convex polygon = 360°.
∴ $360 \div 8 = 45$.
Longer method: The exterior angle is the supplement of the interior angle. The interior angle = (sum of all interior angles) $\div 8 = 6(180) \div 8 = 135$.
$180 - 135 = 45$.

SOLUTIONS - Meet #3 (page 2)

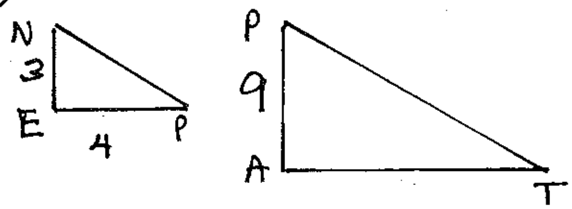
② 

The length of the dotted segment can be found by using it as the hypotenuse of a right triangle as follows:

$$10^2 + 24^2 = x^2$$
$$100 + 576 = x^2$$
$$676 = x^2$$
$$\sqrt{676} = x$$
$$26 = x$$



③ $\triangle PEN$ is similar to $\triangle PAT$. The ratio of similitude is 1:3. Also, via the Pythagorean Theorem, NP = 5.



$\therefore$ AT = 12 and PT = 15.

$$NT = PT - NP = 15 - 5 = 10$$

So, the perimeter of PENTA = PE + EN + NT + AT + AP = 4 + 3 + 10 + 12 + 9 = 38

CATEGORY 3

① 73
② 6
③ 1.2×10^{-1}

① Converting to base 10: $243_{base\,5} = 2(25) + 4(5) + 3(1) = 50 + 20 + 3 = 73$

② This problem translates as follows:

$$235_{base\,7} = \_\_\_\__{base\,3}$$

First convert $235_{base\,7}$ to base 10: $2(49) + 3(7) + 5(1) = 98 + 21 + 5 = 124$

Now convert $124_{base\,10}$ to base 3:
$124 \div 81 = 1$, R 43
$43 \div 27 = 1$, R 16
$16 \div 9 = 1$, R 7
$7 \div 3 = 2$, R 1

$\therefore$ There are: 1 Ricki, 1 Lucy, 1 Desi, 2 Ethels, and 1 Fred, for a total of 6 coins.

③ $$\frac{(3.2 \times 10^{7}) \cdot (4.2 \times 10^{-2})}{(800 \times 10^{-5}) \cdot (0.014 \times 10^{11})}$$

$$= \frac{(32 \cdot 10^{-1} \cdot 10^{7}) \cdot (42 \cdot 10^{-1} \cdot 10^{-2})}{(8 \cdot 10^{2} \cdot 10^{-5}) \cdot (14 \cdot 10^{-3} \cdot 10^{11})}$$

$$= \frac{32 \cdot 42 \cdot 10^{-1} \cdot 10^{7} \cdot 10^{-1} \cdot 10^{-2}}{8 \cdot 14 \cdot 10^{2} \cdot 10^{-5} \cdot 10^{-3} \cdot 10^{11}}$$

$$= \frac{32}{8} \cdot \frac{42}{14} \cdot \frac{10^{3}}{10^{5}}$$

$$= 4 \cdot 3 \cdot 10^{-2}$$

$$= 12 \cdot 10^{-2}$$

$$= 1.2 \cdot 10^{-1}$$

CATEGORY 4

① $15\frac{7}{8}$

② 2

③ 4

① $$2^{3} + 2^{2} + 2^{1} + 2^{0} + 2^{-1} + 2^{-2} + 2^{-3}$$

$$= 8 + 4 + 2 + 1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8}$$

$$= 15\tfrac{7}{8}$$

② $$8^{3} \cdot 4^{-5} \cdot 2^{-6} \cdot 16^{2}$$

It may be easiest to express each quantity as a power of two first:

$$= (2^{3})^{3} \cdot (2^{2})^{-5} \cdot 2^{-6} \cdot (2^{4})^{2}$$

$$= 2^{9} \cdot 2^{-10} \cdot 2^{-6} \cdot 2^{8}$$

$$= 2^{9+(-10)+(-6)+8}$$

$$= 2^{1}$$

$$= 2$$

③ $$\sqrt[5]{32} = 2$$

$$2^{3} = 8$$

$$8^{-2} = \tfrac{1}{64}$$

$$\sqrt[6]{\tfrac{1}{64}} = \tfrac{1}{2}$$

$$(\tfrac{1}{2})^{6} = \tfrac{1}{64}$$

$$\sqrt[3]{\tfrac{1}{64}} = \tfrac{1}{4}$$

$$(\tfrac{1}{4})^{-1} = 4$$

SOLUTIONS - Meet #3 (page 4)

CATEGORY 5

① 19

② 8

③ 3

① $|-6| + 3|7| - 4|-2| + 5|0|$

$= 6 + 3(7) - 4(2) + 5(0)$

$= 6 + 21 - 8 + 0$

$= 19$

② For the expression to be a positive integer, $|x+2|$ would have to be a positive factor of 6, namely 6, 1, 2, or 3. Hence, the following equations:

$|x+2| = 6$, so $x = 4$ or -8

$|x+2| = 1$ so $x = -1$ or -3

$|x+2| = 2$ so $x = 0$ or -4

$|x+2| = 3$ so $x = 1$ or -5

$\therefore$ There are $\underline{\underline{8}}$ values of x.

③ The graph displays $N < -4$, so we need a value of W which gives the same solution for

$$4N - 2(3N-5) + 5W > 33$$
$$4N - 6N + 10 + 5W > 33$$
$$-2N + 10 + 5W > 33$$
$$-2N + 5W > 23$$
$$-2N > 23 - 5W$$
$$N < \frac{23-5W}{-2}$$
$$N < -11\tfrac{1}{2} + 2\tfrac{1}{2}W$$

and since $N < -4$,

then $-11\tfrac{1}{2} + 2\tfrac{1}{2}W = -4$

$$2\tfrac{1}{2}W = -4 + 11\tfrac{1}{2}$$
$$2\tfrac{1}{2}W = 7\tfrac{1}{2}$$
$$W = \frac{7\tfrac{1}{2}}{2\tfrac{1}{2}}$$
$$\therefore \quad W = 3$$

CATEGORY 6

① 156
② 108
③ 8
④ 48
⑤ 126
⑥ 7

① one angle = 180 (# of triangles, created by drawing all diagonals from one vertex) ÷ # of angles

$= \frac{180(13)}{15} = 156^\circ$

② 3^{x+2} can be calculated by multiplying 3^x by 3^2. Substitute 12 for 3^x to get: $3^{x+2} = 3^x \cdot 3^2$
$= 12 \cdot 9$
$= 108$

③ $\frac{H+P}{2} = 27$, so $H+P = 27(2)$, or 54. No algebra is need - just some wise trial + error. Pick reasonable values for H and P to keep their sum fixed at 54, and test their product:

H	P	H+P	H·P	evaluation
20	34	54	680	too low - bring H and P closer together to raise their product.
22	32	54	704	Keep going...
23	31	54	713	ah... just right!

∴ The positive difference of H and P is $31-23 = \underline{\underline{8}}$.

④ A pre-algebra approach: Choose a distance between the two cities which is a common multiple of their speeds, such as 120 miles. The time going from Lynn to Swampscott would be 120 ÷ 40, or 3 hours. The time on the return trip would be 120 ÷ 60, or 2 hours. The average speed for the round trip would be:

$\frac{\text{total distance}}{\text{total time}} = \frac{240 \text{ miles}}{5 \text{ hours}} = 48$ miles per hour.

The answer would be 48 mph, no matter what distance was chosen.

An algebraic approach:

Use the following chart to define variables -

	Rate	Time	Distance
from Lynn to Swampscott	40	T_1	$40T_1$
from Swampscott to Lynn	60	T_2	$60T_2$

SOLUTIONS - Meet #3 (page 6)

The two distances are the same. Solving for T_1:

$$40T_1 = 60T_2$$
$$T_1 = \frac{60T_2}{40}$$
$$T_1 = \frac{3T_2}{2}$$

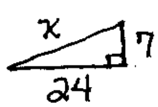
The average speed for the round trip is

$$\frac{\text{total distance}}{\text{total time}}, \text{ or } \quad \frac{40T_1 + 60T_2}{T_1 + T_2}$$

Now substitute $T_1 = \frac{3T_2}{2}$:

$$\frac{40\left(\frac{3T_2}{2}\right) + 60T_2}{\frac{3T_2}{2} + T_2}$$
$$= \frac{60T_2 + 60T_2}{2\frac{1}{2}T_2}$$
$$= \frac{120\cancel{T_2}}{2\frac{1}{2}\cancel{T_2}}$$
$$= 48$$

⑤ Since ΔCAN is a right triangle, CA (the hypotenuse) can be calculated by the Pythagorean Theorem:



$$7^2 + 24^2 = x^2$$
$$49 + 576 = x^2$$
$$625 = x^2$$
$$25 = x$$

Since $\overline{AC}$ is ½ of $\overline{CT}$, CT = 50.

Δ CAN is similar to ΔTAO, so AC : AT = 25 : 75
= 1 : 3

$\therefore \overline{TO} = 3 \cdot \overline{CN} = 3 \cdot 7 = 21$ and $\overline{AO} = 3 \cdot \overline{AN} = 3 \cdot 24 = 72$ and $\overline{NO} = \overline{AO} - \overline{AN} = 72 - 24 = 48$

$\therefore$ The perimeter of NCTO = NC + CT + TO + ON
= 7 + 50 + 21 + 48
= 126 inches.

⑥ How is this done without a calculator? With some partial, but clever factoring, and cancelling:

$$\frac{\frac{D}{C}}{E-(B-C)} \cdot C \cdot \frac{A}{B} \cdot \frac{E}{D}$$

$$= \frac{\frac{48}{8}}{126-(108-8)} \cdot 8 \cdot \frac{156}{108} \cdot \frac{126}{48}$$

$$= \frac{\cancel{6}}{\underset{\cancel{2}}{\cancel{26}}} \cdot \frac{\overset{\cancel{2}}{\cancel{8}}}{1} \cdot \frac{\cancel{12}(\cancel{13})}{\cancel{12}(\cancel{9})} \cdot \frac{\cancel{9}(\cancel{14})^{7}}{\underset{\cancel{2}}{\cancel{12}}(\underset{1}{\cancel{4}})}$$

$$= 7$$

SOLUTIONS - Meet #3 Jan 1998

Mystery
CATEGORY 1

① 49
② 35
③ 276

① Greatest product $= 8 \cdot 8 = 64$
Least product $= 1 \cdot 15 = 15$
Difference $= 64 - 15 = 49$.

② Trial + error : $145 \div 5$, or 29, should be close to how many peanuts Joe ate on the middle (3rd) day. Check: $17+23+29+35+41 = 145$. Yes! ∴ Joe ate 35 peanuts on the 4th day.

③ Number-sense reasoning, using a geometric model:

X	X	X	#23
X	X	X	O
X	X	X	O
X	X	X	O

$\frac{3}{4}$ of savings — remainder

The 16 squares represent \$ in Julie's savings account. The 12 Xs represent the 3/4 of her savings account which she spent to buy the parrot. 3/4 of the remainder is represented by the 3 Os. #23 is left, which is 1/16 of what was originally in Julie's savings.

∴ Multiply 23 by 12 (the # of Xs) = #276.

Algebraic reasoning:

Let x = # in savings account

then $x - 3/4x - 3/4(x - 3/4x) = 23$

$x - 3/4x - 3/4x + 9/16x = 23$

$1/16\, x = 23$

$16 \cdot 1/16\, x = 16 \cdot 23$

(# originally in savings) $x = 368$

(cost of parrot) $3/4\, x = 276$

Geometry January 1998

CATEGORY 2

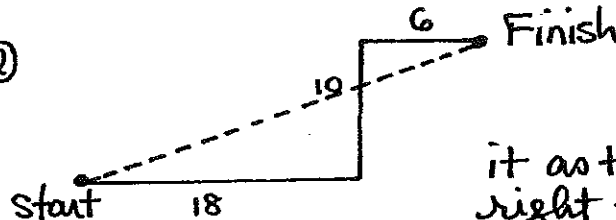
① 45

② 26

③ 38

① Quickest method : The sum of the exterior angles of any convex polygon = 360°.
$\therefore$ 360 ÷ 8 = 45.
Longer method : The exterior angle is the supplement of the interior angle. The interior angle = (sum of all interior angles) ÷ 8 = 6(180) ÷ 8 = 135.
180 − 135 = 45.

②



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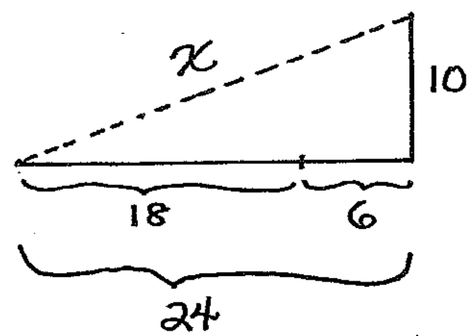
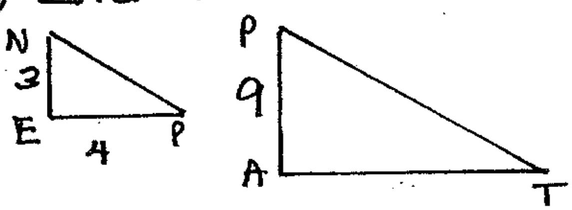


③ △PEN is similar to △PAT. The ratio of similitude is 1:3. Also, via the Pythagorean Theorem, NP = 5.



$\therefore$ AT = 12 and PT = 15.
NT = PT − NP
= 15 − 5
= 10

So, the perimeter of PENTA = PE + EN + NT + AT + AP
= 4 + 3 + 10 + 12 + 9
= 38